European Barrier Call Options

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Let S be the price of the underlying and σ be its instantaneous volatility.

Let C be the European call price and r be the instantaneous risk-free rate.

Let T be the time to maturity and K be the exercise price of the call.

Let V be the European down-and-out call price and $\bar{V}$ be the European down-and-in call price corresponding to the call C, with common barrier price B.

The call is such that $C(S,T) = \max(S_t - K, 0)$, and there is a closed-form solution given by the Black and Scholes formula.

The down-and-out call is such that $\{S_t > B \forall t \in [0,T]\} \Rightarrow V(S,T) = \max(S_t - K, 0)$ and $V(S,T) = 0$ otherwise.

The down-and-in call is such that $\{\exists t \in [0,T] \ni S_t \leq B\} \Rightarrow \bar{V}(S,T) = \max(S_t - K, 0)$ and $\bar{V}(S,T) = 0$ otherwise.

Since $S_t > B \forall t \in [0,T]$ and $\exists t \in [0,T] \ni S_t \leq B$ are mutually exclusive and together exhaustive, a down-and-in call is complementary to a down-and-out call, so that: $\bar{V}(S,T) + V(S,T) = C(S,T)$.

The closed form solution for the down-and-out call is given by the following formula:

$$V(S,T) = C(S,T) - \left(\frac{B}{S}\right)^{\alpha} C\left(\frac{B^2}{S}, T\right)$$

where $\alpha = \frac{2r}{\sigma^2} - 1$, so that the closed form solution for the down-and-in call is given by the following formula:

$$\bar{V}(S,T) = \left(\frac{B}{S}\right)^{\alpha} C\left(\frac{B^2}{S}, T\right)$$

References

[1] Franke, J., Härdle, W.K., Hafner, C.M., Statistics of Financial Markets - An Introduction, Second Edition, Springer, 2008, pp.146-8